

# MAXIMAL AND AREA INTEGRAL CHARACTERIZATIONS OF BERGMAN SPACES IN THE UNIT BALL OF $\mathbb{C}^n$

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**ABSTRACT.** In this paper, we present maximal and area integral characterizations of Bergman spaces in the unit ball of  $\mathbb{C}^n$ . The characterizations are in terms of maximal functions and area integral functions on Bergman balls involving the radial derivative, the complex gradient, and the invariant gradient. As an application, we obtain new maximal and area integral characterizations of Besov spaces. A special case of this is new characterizations of Hardy-Sobolev spaces involving maximal and area integral functions defined on Bergman balls in the unit ball of  $\mathbb{C}^n$ . Moreover, we give a real-variable atomic decomposition of Bergman spaces with respect to Carleson tubes.

## 1. INTRODUCTION AND MAIN RESULTS

Let  $\mathbb{C}$  denote the set of complex numbers. Throughout the paper we fix a positive integer  $n$ , and let

$$\mathbb{C}^n = \mathbb{C} \times \cdots \times \mathbb{C}$$

denote the Euclidean space of complex dimension  $n$ . Addition, scalar multiplication, and conjugation are defined on  $\mathbb{C}^n$  componentwise. For  $z = (z_1, \dots, z_n)$  and  $w = (w_1, \dots, w_n)$  in  $\mathbb{C}^n$ , we write

$$\langle z, w \rangle = z_1 \overline{w}_1 + \cdots + z_n \overline{w}_n,$$

where  $\overline{w}_k$  is the complex conjugate of  $w_k$ . We also write

$$|z| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}.$$

The open unit ball in  $\mathbb{C}^n$  is the set

$$\mathbb{B}_n = \{z \in \mathbb{C}^n : |z| < 1\}.$$

The boundary of  $\mathbb{B}_n$  will be denoted by  $\mathbb{S}_n$  and is called the unit sphere in  $\mathbb{C}^n$ , i.e.,

$$\mathbb{S}_n = \{z \in \mathbb{C}^n : |z| = 1\}.$$

Also, we denote by  $\overline{\mathbb{B}}_n$  the closed unit ball, i.e.,

$$\overline{\mathbb{B}}_n = \{z \in \mathbb{C}^n : |z| \leq 1\} = \mathbb{B}_n \cup \mathbb{S}_n.$$

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*2010 Mathematics Subject Classification:* 32A36, 32A50.

*Key words:* Bergman space, Bergman metric, maximal function, area integral function, atomic decomposition.

The automorphism group of  $\mathbb{B}^n$ , denoted by  $\text{Aut}(\mathbb{B}^n)$ , consists of all bi-holomorphic mappings of  $\mathbb{B}^n$ . Traditionally, bi-holomorphic mappings are also called automorphisms.

For  $\alpha \in \mathbb{R}$ , the weighted Lebesgue measure  $dv_\alpha$  on  $\mathbb{B}_n$  is defined by

$$dv_\alpha(z) = c_\alpha(1 - |z|^2)^\alpha dv(z)$$

where  $c_\alpha = 1$  for  $\alpha \leq -1$  and  $c_\alpha = \Gamma(n + \alpha + 1)/[n!\Gamma(\alpha + 1)]$  if  $\alpha > -1$ , which is a normalizing constant so that  $dv_\alpha$  is a probability measure on  $\mathbb{B}_n$ . In the case of  $\alpha = -(n + 1)$  we denote the resulting measure by

$$d\tau(z) = \frac{dv}{(1 - |z|^2)^{n+1}},$$

and call it the invariant measure on  $\mathbb{B}^n$ , since  $d\tau = d\tau \circ \varphi$  for any automorphism  $\varphi$  of  $\mathbb{B}^n$ .

For  $\alpha > -1$  and  $p > 0$ , the (weighted) Bergman space  $\mathcal{A}_\alpha^p$  consists of holomorphic functions  $f$  in  $\mathbb{B}_n$  with

$$\|f\|_{p,\alpha} = \left( \int_{\mathbb{B}_n} |f(z)|^p dv_\alpha(z) \right)^{1/p} < \infty,$$

where the weighted Lebesgue measure  $dv_\alpha$  on  $\mathbb{B}_n$  is defined by

$$dv_\alpha(z) = c_\alpha(1 - |z|^2)^\alpha dv(z)$$

and  $c_\alpha = \Gamma(n + \alpha + 1)/[n!\Gamma(\alpha + 1)]$  is a normalizing constant so that  $dv_\alpha$  is a probability measure on  $\mathbb{B}_n$ . Thus,

$$\mathcal{A}_\alpha^p = \mathcal{H}(\mathbb{B}_n) \cap L^p(\mathbb{B}_n, dv_\alpha),$$

where  $\mathcal{H}(\mathbb{B}_n)$  is the space of all holomorphic functions in  $\mathbb{B}_n$ . When  $\alpha = 0$  we simply write  $\mathcal{A}^p$  for  $\mathcal{A}_0^p$ . These are the usual Bergman spaces. Note that for  $1 \leq p < \infty$ ,  $\mathcal{A}_\alpha^p$  is a Banach space under the norm  $\|\cdot\|_{p,\alpha}$ . If  $0 < p < 1$ , the space  $\mathcal{A}_\alpha^p$  is a quasi-Banach space with  $p$ -norm  $\|f\|_{p,\alpha}^p$ .

Recall that  $D(z, \gamma)$  denotes the Bergman metric ball at  $z$

$$D(z, \gamma) = \{w \in \mathbb{B}_n : \beta(z, w) < \gamma\}$$

with  $\gamma > 0$ , where  $\beta$  is the Bergman metric on  $\mathbb{B}_n$ . It is known that

$$\beta(z, w) = \frac{1}{2} \log \frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|}, \quad z, w \in \mathbb{B}_n,$$

whereafter  $\varphi_z$  is the bijective holomorphic mapping in  $\mathbb{B}_n$ , which satisfies  $\varphi_z(0) = z$ ,  $\varphi_z(z) = 0$  and  $\varphi_z \circ \varphi_z = id$ .

As is well known, maximal functions play a crucial role in the real-variable theory of Hardy spaces (cf. [11]). In this paper, we first establish a maximal-function characterization for the Bergman spaces. To this end, we define for each  $\gamma > 0$  and  $f \in \mathcal{H}(\mathbb{B}_n)$ :

$$(1.1) \quad (M_\gamma f)(z) = \sup_{w \in D(z, \gamma)} |f(w)|, \quad \forall z \in \mathbb{B}_n.$$

We begin with the following simple result.

**Theorem 1.1.** *Suppose  $\gamma > 0$  and  $\alpha > -1$ . Let  $0 < p < \infty$ . Then for any  $f \in \mathcal{H}(\mathbb{B}_n)$ ,  $f \in \mathcal{A}_\alpha^p$  if and only if  $M_\gamma f \in L^p(\mathbb{B}_n, dv_\alpha)$ . Moreover,*

$$(1.2) \quad \|f\|_{p,\alpha} \approx \|M_\gamma f\|_{p,\alpha},$$

where “ $\approx$ ” depends only on  $\gamma, \alpha, p$ , and  $n$ .

The norm appearing on the right-hand side of (1.2) can be viewed an analogue of the so-called nontangential maximal function in Hardy spaces. The proof of Theorem 1.1 is fairly elementary (see §2), using some basic facts and estimates on the Bergman balls.

In order to state the real-variable area integral characterizations of the Bergman spaces, we require some more notation. For any  $f \in \mathcal{H}(\mathbb{B}_n)$  and  $z = (z_1, \dots, z_n) \in \mathbb{B}_n$  we define

$$\mathcal{R}f(z) = \sum_{k=1}^n z_k \frac{\partial f(z)}{\partial z_k}$$

and call it the radial derivative of  $f$  at  $z$ . The complex and invariant gradients of  $f$  at  $z$  are respectively defined as

$$\nabla f(z) = \left( \frac{\partial f(z)}{\partial z_1}, \dots, \frac{\partial f(z)}{\partial z_n} \right) \text{ and } \tilde{\nabla} f(z) = \nabla(f \circ \varphi_z)(0).$$

Now, for fixed  $1 < q < \infty$  and  $\gamma > 0$ , we define for each  $f \in \mathcal{H}(\mathbb{B}_n)$  and  $z \in \mathbb{B}_n$ :

(1) The radial area function

$$A_{\mathcal{R}}^{\gamma,q}(f)(z) = \left( \int_{D(z,\gamma)} |(1-|w|^2)\mathcal{R}f(w)|^q d\tau(w) \right)^{\frac{1}{q}}.$$

(2) The complex gradient area function

$$A_{\nabla}^{\gamma,q}(f)(z) = \left( \int_{D(z,\gamma)} |(1-|w|^2)\nabla f(w)|^q d\tau(w) \right)^{\frac{1}{q}}.$$

(3) The invariant gradient area function

$$A_{\tilde{\nabla}}^{\gamma,q}(f)(z) = \left( \int_{D(z,\gamma)} |\tilde{\nabla} f(w)|^q d\tau(w) \right)^{\frac{1}{q}}.$$

We state another main result of this paper as follows.

**Theorem 1.2.** *Suppose  $1 < q < \infty, \gamma > 0$ , and  $\alpha > -1$ . Let  $0 < p < \infty$ . Then, for any  $f \in \mathcal{H}(\mathbb{B}_n)$  the following conditions are equivalent:*

- (a)  $f \in \mathcal{A}_\alpha^p$ .
- (b)  $A_{\mathcal{R}}^{\gamma,q}(f)$  is in  $L^p(\mathbb{B}_n, dv_\alpha)$ .
- (c)  $A_{\nabla}^{\gamma,q}(f)$  is in  $L^p(\mathbb{B}_n, dv_\alpha)$ .
- (d)  $A_{\tilde{\nabla}}^{\gamma,q}(f)$  is in  $L^p(\mathbb{B}_n, dv_\alpha)$ .

Moreover, the quantities

$$\|A_{\mathcal{R}}^{\gamma,q}(f)\|_{p,\alpha}, \|A_{\nabla}^{\gamma,q}(f)\|_{p,\alpha}, \|A_{\nabla^*}^{\gamma,q}(f)\|_{p,\alpha},$$

are all comparable to  $\|f - f(0)\|_{p,\alpha}$ , where the comparable constants depend only on  $q, \gamma, \alpha, p$ , and  $n$ .

For  $0 < p < \infty$  and  $-\infty < \alpha < \infty$  we fix a nonnegative integer  $k$  with  $pk + \alpha > -1$  and define the so-called Bergman space  $\mathcal{A}_\alpha^p$  introduced in [14] as the space of all  $f \in \mathcal{H}(\mathbb{B}_n)$  such that  $(1 - |z|^2)^k \mathcal{R}^k f \in L^p(\mathbb{B}_n, dv_\alpha)$ . One then easily observes that  $\mathcal{A}_\alpha^p$  is independent of the choice of  $k$  and consistent with the traditional definition when  $\alpha > -1$ . Let  $N$  be the smallest nonnegative integer such that  $pN + \alpha > -1$  and define

$$(1.3) \quad \|f\|_{p,\alpha} = |f(0)| + \left( \int_{\mathbb{B}_n} (1 - |z|^2)^{pN} |\mathcal{R}^N f(z)|^p dv_\alpha(z) \right)^{\frac{1}{p}}, \quad f \in \mathcal{A}_\alpha^p.$$

Equipped with (1.3),  $\mathcal{A}_\alpha^p$  becomes a Banach space when  $p \geq 1$  and a quasi-Banach space for  $0 < p < 1$ .

It is known that the family of the generalized Bergman spaces  $\mathcal{A}_\alpha^p$  covers most of the spaces of holomorphic functions in the unit ball of  $\mathbb{C}^n$ , such as the classical diagonal Besov space  $B_p^s$  and the Sobolev space  $W_{k,\beta}^p$  (e.g., [5]), which has been extensively studied before in the literature under different names (e.g., see [14] for an overview). We refer to Arcozzi-Rochberg-Sawyer [3, 4], Tchoundja [12] and Volberg-Wick [13] for some recent results on such Besov spaces and more references.

As an application of Theorems 1.1 and 1.2, we obtain new maximal and area integral characterizations of such Besov spaces as follows.

**Corollary 1.1.** *Suppose  $\gamma > 0$  and  $\alpha \in \mathbb{R}$ . Let  $0 < p < \infty$  and  $k$  be a nonnegative integer such that  $pk + \alpha > -1$ . Then for any  $f \in \mathcal{H}(\mathbb{B}_n)$ ,  $f \in \mathcal{A}_\alpha^p$  if and only if  $M_\gamma^{(k)}(f) \in L^p(\mathbb{B}_n, dv_\alpha)$ , where*

$$(1.4) \quad M_\gamma^{(k)}(f)(z) = \sup_{w \in D(z,\gamma)} |(1 - |w|^2)^k \mathcal{R}^k f(w)|, \quad z \in \mathbb{B}_n.$$

Moreover,

$$(1.5) \quad \|f - f(0)\|_{p,\alpha} \approx \|M_\gamma(\mathcal{R}^k f)\|_{p,\alpha},$$

where “ $\approx$ ” depends only on  $\gamma, \alpha, p, k$ , and  $n$ .

**Corollary 1.2.** *Suppose  $1 < q < \infty, \gamma > 0$  and  $\alpha \in \mathbb{R}$ . Let  $0 < p < \infty$  and  $k$  be a nonnegative integer such that  $pk + \alpha > -1$ . Then for any  $f \in \mathcal{H}(\mathbb{B}_n)$ ,  $f \in \mathcal{A}_\alpha^p$  if and only if  $A_{\mathcal{R}^{k+1}}^{\gamma,q}(f)$  is in  $L^p(\mathbb{B}_n, dv_\alpha)$ , where*

$$(1.6) \quad A_{\mathcal{R}^k}^{\gamma,q}(f)(z) = \left( \int_{D(z,\gamma)} |(1 - |w|^2)^k \mathcal{R}^k f(w)|^q d\tau(w) \right)^{\frac{1}{q}}.$$

Moreover,

$$(1.7) \quad \|f - f(0)\|_{p,\alpha} \approx \|A_{\mathcal{R}^{k+1}}^{\gamma,q}(f)\|_{p,\alpha},$$

where “ $\approx$ ” depends only on  $q, \gamma, \alpha, p, k$ , and  $n$ .

To prove Corollaries 1.1 and 1.2, one merely notices that  $f \in \mathcal{A}_\alpha^p$  if and only if  $\mathcal{R}^k f \in L^p(\mathbb{B}_n, dv_{\alpha+pk})$  and applies Theorems 1.1 and 1.2 respectively to  $\mathcal{R}^k f$  with the help of Lemma 2.1 below.

There are various characterizations for  $B_p^s$  or  $W_{k,\beta}^p$  involving complex-variable quantities in terms of fractional differential operators and in terms of higher order (radical) derivatives and/or complex and invariant gradients, for a review and details see [14] and references therein. However, Corollaries 1.1 and 1.2 can be considered as a unified characterization for such spaces involving real-variable quantities.

In particular,  $\mathcal{H}_s^p = \mathcal{A}_\alpha^p$  with  $\alpha = -2s - 1$ , where  $\mathcal{H}_s^p$  is the Hardy-Sobolev space defined as the set

$$\left\{ f \in \mathcal{H}(\mathbb{B}_n) : \|f\|_{\mathcal{H}_s^p}^p = \sup_{0 < r < 1} \int_{\mathbb{S}_n} |(I + \mathcal{R})^s f(r\zeta)|^p d\sigma(\zeta) < \infty \right\}.$$

Here,

$$(I + \mathcal{R})^s f = \sum_{k=0}^{\infty} (1+k)^s f_k$$

if  $f = \sum_{k=0}^{\infty} f_k$  is the homogeneous expansion of  $f$ . There are several real-variable characterizations of the Hardy-Sobolev spaces obtained by Ahern and Bruna [1] (see also [2]). These characterizations are in terms of maximal and area integral functions on the admissible approach region

$$D_\alpha(\eta) = \left\{ z \in \mathbb{B}_n : |1 - \langle z, \eta \rangle| < \frac{\alpha}{2}(1 - |z|^2) \right\}, \quad \eta \in \mathbb{S}_n, \alpha > 1.$$

Evidently, Corollaries 1.1 and 1.2 present new maximal and area integral descriptions of the Hardy-Sobolev spaces in terms of the Bergman metric. A special case of this is a characterization of the usual Hardy space  $\mathcal{H}^p = \mathcal{A}_{-1}^p$  itself.

The rest of the paper is organized as follows. In Section 2 we will prove Theorems 1.1 and 1.2 using some elementary facts about Bergman metric and Bergman kernel functions. In particular, we will prove Theorem 1.2 in the case of  $0 < p \leq 1$  by using “complex-variable” atom decomposition for Bergman spaces due to Coifman and Rochberg [8]. In Section 3, we will prove an atomic decomposition of  $\mathcal{A}_\alpha^1$  with respect to Carleson tubes, which is used to give a real-variable proof of Theorem 1.2 in the case  $p = 1$  in Section 4.

In what follows,  $C$  always denotes a constant depending (possibly) on  $n, q, p, \gamma$  or  $\alpha$  but not on  $f$ , which may be different in different places. For two nonnegative (possibly infinite) quantities  $X$  and  $Y$ , by  $X \lesssim Y$  we mean that there exists a constant  $C > 0$  such that  $X \leq CY$  and by  $X \approx Y$  that  $X \lesssim Y$  and  $Y \lesssim X$ . Any notation and terminology not otherwise explained, are as used in [15] for spaces of holomorphic functions in the unit ball of  $\mathbb{C}^n$ .

## 2. PROOFS OF THEOREMS 1.1 AND 1.2: COMPLEX METHODS

For the sake of convenience, we collect some elementary facts on the Bergman metric and holomorphic functions in the unit ball of  $\mathbb{C}^n$  as follows.

**Lemma 2.1.** (cf. Lemma 2.20 in [15]) *For each  $\gamma > 0$ ,*

$$1 - |a|^2 \approx 1 - |z|^2 \approx |1 - \langle a, z \rangle|$$

*for all  $a$  and  $z$  in  $\mathbb{B}_n$  with  $\beta(a, z) < \gamma$ .*

**Lemma 2.2.** (cf. Lemma 2.24 in [15]) *Suppose  $\gamma > 0, p > 0$ , and  $\alpha > -1$ . Then there exists a constant  $C > 0$  such that for any  $f \in \mathcal{H}(\mathbb{B}_n)$ ,*

$$|f(z)|^p \leq \frac{C}{(1 - |z|^2)^{n+1+\alpha}} \int_{D(z, \gamma)} |f(w)|^p dv_\alpha(w), \quad \forall z \in \mathbb{B}_n.$$

**Lemma 2.3.** (cf. Lemma 2.27 in [15]) *For each  $\gamma > 0$ ,*

$$|1 - \langle z, u \rangle| \approx |1 - \langle z, v \rangle|$$

*for all  $z$  in  $\bar{\mathbb{B}}_n$  and  $u, v$  in  $\mathbb{B}_n$  with  $\beta(u, v) < \gamma$ .*

We first prove Theorem 1.1. We need the following result (cf. Lemma 5 in [6]).

**Lemma 2.4.** *For fixed  $\gamma > 0$ , there exist a positive integer  $N$  and a sequence  $\{a_k\}$  in  $\mathbb{B}_n$  such that*

- (1)  $\mathbb{B}_n = \cup_k D(a_k, \gamma)$ , and
- (2) *each  $z \in \mathbb{B}_n$  belongs to at most  $N$  of the sets  $D(a_k, 3\gamma)$ .*

*Proof of Theorem 1.1.* Let  $p > 0$ . By Lemmas 2.4, 2.2, and 2.1, we have

$$\begin{aligned} & \int_{\mathbb{B}_n} |M_\gamma(f)(z)|^p dv_\alpha(z) \\ & \leq \sum_k \int_{D(a_k, \gamma)} |M_\gamma(f)(z)|^p dv_\alpha(z) \\ & = \sum_k \int_{D(a_k, \gamma)} \sup_{w \in D(z, \gamma)} |f(w)|^p dv_\alpha(z) \\ & \lesssim \sum_k \int_{D(a_k, \gamma)} \sup_{w \in D(z, \gamma)} \frac{1}{(1 - |w|^2)^{n+1+\alpha}} \int_{D(w, \gamma)} |f(u)|^p dv_\alpha(u) dv_\alpha(z) \\ & \lesssim \sum_k \int_{D(a_k, \gamma)} \left( \frac{1}{(1 - |a_k|^2)^{n+1+\alpha}} \int_{D(a_k, 3\gamma)} |f(u)|^p dv_\alpha(u) \right) dv_\alpha(z) \\ & \lesssim \sum_k \int_{D(a_k, 3\gamma)} |f(u)|^p dv_\alpha(u) \\ & \lesssim N \int_{\mathbb{B}_n} |f(u)|^p dv_\alpha(u) \end{aligned}$$

where  $N$  is the constant in Lemma 2.4 depending only on  $\gamma$  and  $n$ .  $\square$

Let  $1 \leq p < \infty$  and let  $E$  be a complex Banach space. We write  $L_\alpha^p(\mathbb{B}_n, E)$  for the Banach space of strongly measurable  $E$ -valued functions on  $\mathbb{B}_n$  such that

$$\left( \int_{\mathbb{B}_n} \|f(z)\|_E^p dv_\alpha(z) \right)^{\frac{1}{p}} < \infty.$$

Recall that  $f : \mathbb{B}_n \mapsto E$  is said to be holomorphic if for each  $x^* \in E^*$ ,  $x^*f$  is holomorphic in  $\mathbb{B}_n$ . It is known that (for example, see [10]) if  $f$  is holomorphic in this weak sense, then it is holomorphic in the stronger sense that  $f$  is the sum of a power series

$$f(z) = \sum_{J \in \mathbb{N}_0^n} x_J z^J, \quad z \in \mathbb{B}_n,$$

where  $x_J \in E$ . (As usual,  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ .) The class of all such functions is denoted by  $\mathcal{H}(\mathbb{B}_n, E)$ . We write  $\mathcal{A}_\alpha^p(\mathbb{B}_n, E)$  for the class of weighted  $E$ -valued Bergman space of functions  $f \in \mathcal{H}(\mathbb{B}_n, E) \cap L_\alpha^p(\mathbb{B}_n, E)$ .

Also, we define  $\mathcal{B}(\mathbb{B}_n, E)$  as the space of all  $f \in \mathcal{H}(\mathbb{B}_n, E)$  so that

$$\|f\|_{\mathcal{B}} = \sup_{z \in \mathbb{B}_n} \|\tilde{\nabla} f(z)\|_E < \infty.$$

$\mathcal{B}(\mathbb{B}_n, E)$  with the norm  $\|f\| = \|f(0)\|_E + \|f\|_{\mathcal{B}}$  is the  $E$ -valued Bloch space. When  $E = \mathbb{C}$ , we simply write  $\mathcal{B} = \mathcal{B}(\mathbb{B}_n, \mathbb{C})$ .

Then, by merely repeating the proof of the scalar case (e.g., Theorem 3.25 in [15]), we have the following interpolation result.

**Lemma 2.5.** *Suppose  $\alpha > -1$  and*

$$\frac{1}{p} = \frac{1-\theta}{p'}$$

*for  $0 < \theta < 1$  and  $1 \leq p' < \infty$ . Then*

$$\left[ \mathcal{A}_\alpha^{p'}(\mathbb{B}_n, E), \mathcal{B}(\mathbb{B}_n, E) \right]_\theta = \mathcal{A}_\alpha^p(\mathbb{B}_n, E)$$

*with equivalent norms.*

Moreover, to prove Theorem 1.2 for the case  $0 < p \leq 1$ , we will use atom decomposition for Bergman spaces due to Coifman and Rochberg [8] (see also [15], Theorem 2.30) as follows.

**Proposition 2.1.** *Suppose  $p > 0$ ,  $\alpha > -1$ , and  $b > n \max\{1, 1/p\} + (\alpha + 1)/p$ . Then there exists a sequence  $\{a_k\}$  in  $\mathbb{B}_n$  such that  $\mathcal{A}_\alpha^p$  consists exactly of functions of the form*

$$f(z) = \sum_{k=1}^{\infty} c_k \frac{(1 - |a_k|^2)^{(pb-n-1-\alpha)/p}}{(1 - \langle z, a_k \rangle)^b}, \quad z \in \mathbb{B}_n,$$

where  $\{c_k\}$  belongs to the sequence space  $\ell^p$  and the series converges in the norm topology of  $\mathcal{A}_\alpha^p$ . Moreover,

$$\int_{\mathbb{B}_n} |f(z)|^p dv_\alpha(z) \approx \inf \left\{ \sum_k |c_k|^p \right\},$$

where the infimum runs over all the above decompositions.

Also, we need a characterization of Carleson type measures for Bergman spaces as follows, which can be found in [14], Theorem 45.

**Proposition 2.2.** *Suppose  $n + 1 + \alpha > 0$  and  $\mu$  is a positive Borel measure on  $\mathbb{B}_n$ . Then, there exists a constant  $C > 0$  such that*

$$\mu(Q_r(\zeta)) \leq Cr^{2(n+1+\alpha)}, \quad \forall \zeta \in \mathbb{S}_n \text{ and } r > 0,$$

if and only if for each  $s > 0$  there exists a constant  $C > 0$  such that

$$\int_{\mathbb{B}_n} \frac{(1 - |z|^2)^s}{|1 - \langle z, w \rangle|^{n+1+\alpha+s}} d\mu(w) \leq C$$

for all  $z \in \mathbb{B}_n$ .

We are now ready to prove Theorem 1.2. Note that for any  $f \in \mathcal{H}(\mathbb{B}_n)$ ,

$$(1 - |z|^2)|\mathcal{R}f(z)| \leq (1 - |z|^2)|\nabla f(z)| \leq |\tilde{\nabla} f(z)|, \quad \forall z \in \mathbb{B}_n.$$

(e.g., Lemma 2.14 in [15].) We have that (d) implies (c), and (c) implies (b) in Theorem 1.2. Then, it remains to prove that (b) implies (a), and (a) implies (d).

*Proof of (b)  $\Rightarrow$  (a).* Since  $\mathcal{R}f(z)$  is holomorphic, by Lemma 2.2 we have

$$\begin{aligned} |\mathcal{R}f(z)|^q &\leq \frac{C}{(1 - |z|^2)^{n+1}} \int_{D(z, \gamma)} |\mathcal{R}f(w)|^q dv(w) \\ &\leq C_\gamma \int_{D(z, \gamma)} |\mathcal{R}f(w)|^q d\tau(w). \end{aligned}$$

Then,

$$\begin{aligned} (1 - |z|^2)|\mathcal{R}f(z)| &\leq C(1 - |z|^2) \left( \int_{D(z, \gamma)} |\mathcal{R}f(w)|^q d\tau(w) \right)^{\frac{1}{q}} \\ &\leq C_\gamma \left( \int_{D(z, \gamma)} |(1 - |w|^2)\mathcal{R}f(w)|^q d\tau(w) \right)^{\frac{1}{q}} \\ &= C_\gamma A_{\mathcal{R}}^{\gamma, q}(f)(z). \end{aligned}$$

Hence, for any  $p > 0$ , if  $A_{\mathcal{R}}^{\gamma, q}(f) \in L^p(\mathbb{B}_n, dv_\alpha)$  then  $(1 - |z|^2)|\mathcal{R}f(z)|$  is in  $L^p(\mathbb{B}_n, dv_\alpha)$ , which implies that  $f \in \mathcal{A}_\alpha^p$  (e.g., Theorem 2.16 in [15]).  $\square$



The proof of (a)  $\Rightarrow$  (d) is divided into two steps. At first we prove the case of  $0 < p \leq 1$  using the atomic decomposition, then we prove the generic case via complex interpolation.

*Proof of (a)  $\Rightarrow$  (d) for  $0 < p \leq 1$ .* To this end, we write

$$f_k(z) = \frac{(1 - |a_k|^2)^{(pb-n-1-\alpha)/p}}{(1 - \langle z, a_k \rangle)^b}.$$

An immediate computation yields that

$$\nabla f_k(z) = \frac{b\bar{a}_k(1 - |a_k|^2)^{(pb-n-1-\alpha)/p}}{(1 - \langle z, a_k \rangle)^{b+1}}$$

and

$$\mathcal{R}f_k(z) = \frac{b\langle z, a_k \rangle(1 - |a_k|^2)^{(pb-n-1-\alpha)/p}}{(1 - \langle z, a_k \rangle)^{b+1}}$$

Then we have

$$\begin{aligned} |\tilde{\nabla} f_k(z)|^2 &= (1 - |z|^2)(|\nabla f_k(z)|^2 - |\mathcal{R}f_k(z)|^2) \\ &= b^2(1 - |z|^2)(1 - |a_k|^2)^{2(pb-n-1-\alpha)/p} \frac{|a_k|^2 - |\langle z, a_k \rangle|^2}{|1 - \langle z, a_k \rangle|^{2(b+1)}}. \end{aligned}$$

By Lemmas 2.1 and 2.3 one has

$$\begin{aligned} A_{\tilde{\nabla}}^{\gamma,q}(f_k)(z) &= \left( \int_{D(z,\gamma)} |\tilde{\nabla} f_k(w)|^q d\tau(w) \right)^{\frac{1}{q}} \\ &\leq b(1 - |a_k|^2)^{(pb-n-1-\alpha)/p} \left( \int_{D(z,\gamma)} \frac{1}{|1 - \langle w, a_k \rangle|^{qb}} d\tau(w) \right)^{\frac{1}{q}} \\ &\leq \frac{C_\gamma b(1 - |a_k|^2)^{(pb-n-1-\alpha)/p}}{|1 - \langle z, a_k \rangle|^b}, \end{aligned}$$

where we have used the fact  $v(D(z, \gamma)) \approx (1 - |z|^2)^{n+1}$ . Note that  $v_\alpha(Q_r) \approx r^{2(n+1+\alpha)}$  (e.g., Corollary 5.24 in [15]), by Proposition 2.2 we have

$$\int_{\mathbb{B}_n} |A_{\tilde{\nabla}}^{\gamma,q}(f_k)(z)|^p dv_\alpha(z) \leq Cb^p \int_{\mathbb{B}_n} \frac{(1 - |a_k|^2)^{(pb-n-1-\alpha)}}{|1 - \langle z, a_k \rangle|^{pb}} dv_\alpha(z) \leq C_{p,\alpha}.$$

Hence, for  $0 < p \leq 1$  we have for  $f = \sum_{k=1}^{\infty} c_k f_k$  with  $\sum_k |c_k|^p < \infty$ ,

$$\int_{\mathbb{B}_n} |A_{\tilde{\nabla}}^{\gamma,q}(f)(z)|^p dv_\alpha \leq \sum_{k=1}^{\infty} |c_k|^p \int_{\mathbb{B}_n} |A_{\tilde{\nabla}}^{\gamma,q}(f_k)(z)|^p dv_\alpha \leq C_{p,\alpha} \sum_{k=1}^{\infty} |c_k|^p.$$

This concludes that

$$\int_{\mathbb{B}_n} |A_{\tilde{\nabla}}^{\gamma,q}(f)(z)|^p dv_\alpha \leq C_{p,\alpha} \inf \left\{ \sum_{k=1}^{\infty} |c_k|^p \right\} \leq C_{p,\alpha} \int_{\mathbb{B}_n} |f(z)|^p dv_\alpha(z).$$

The proof is complete.  $\square$

*Proof of (a)  $\Rightarrow$  (d) for  $p > 1$ .* Set  $E = L^q(\mathbb{B}_n, \chi_{D(0,\gamma)} d\tau; \mathbb{C}^n)$ . Consider the operator

$$T(f)(z, w) = (\tilde{\nabla} f)(\varphi_z(w)), \quad f \in \mathcal{H}(\mathbb{B}_n).$$

Note that  $\varphi_z(D(0, \gamma)) = D(z, \gamma)$  and the measure  $d\tau$  is invariant under any automorphism of  $\mathbb{B}_n$  (cf. Proposition 1.13 in [15]), we have

$$\begin{aligned} \|T(f)(z)\|_E &= \left( \int_{\mathbb{B}_n} |(\tilde{\nabla} f)(\varphi_z(w))|^q \chi_{D(0,\gamma)}(w) d\tau(w) \right)^{\frac{1}{q}} \\ &= \left( \int_{\mathbb{B}_n} |\tilde{\nabla} f(w)|^q \chi_{D(z,\gamma)}(w) d\tau(w) \right)^{\frac{1}{q}} \\ &= A_{\tilde{\nabla}}^{\gamma,q}(f)(z). \end{aligned}$$

On the other hand,

$$A_{\tilde{\nabla}}^{\gamma,q}(f)(z) \leq [C_\gamma(1 - |z|^2)^{-n-1} v(D(z, \gamma))]^{\frac{1}{2}} \|f\|_{\mathcal{B}} \leq C \|f\|_{\mathcal{B}}.$$

Then, we conclude that  $T$  is bounded from  $\mathcal{B}$  into  $\mathcal{B}(\mathbb{B}_n, E)$ . Thus, applying Lemma 2.5 to this fact with the case of  $p = 1$  proved above yields that  $T$  is bounded from  $\mathcal{A}_\alpha^p$  into  $\mathcal{A}_\alpha^p(\mathbb{B}_n, E)$  for any  $1 < p < \infty$ , i.e.,

$$\|A_{\tilde{\nabla}}^{\gamma,q}(f)\|_{p,\alpha} \leq C \|f\|_{p,\alpha}, \quad \forall f \in \mathcal{A}_\alpha^p,$$

where  $C$  depends only on  $q, \gamma, n, p$ , and  $\alpha$ . The proof is complete.  $\square$

**Remark 2.1.** From the proofs of that (b)  $\implies$  (a) and that (a)  $\Rightarrow$  (d) for  $p > 1$  we find that Theorem 1.2 still holds true for the Bloch space. That is, for any  $f \in \mathcal{H}(\mathbb{B}_n)$ ,  $f \in \mathcal{B}$  if and only if one (or equivalent, all) of  $A_{\mathcal{R}}^{\gamma,q}(f)$ ,  $A_{\tilde{\nabla}}^{\gamma,q}(f)$ , and  $A_{\tilde{\nabla}}^{\gamma,q}(f)$  is (or are) in  $L^\infty(\mathbb{B}_n)$ . Moreover,

$$(2.1) \quad \|f\|_{\mathcal{B}} \approx \|A_{\mathcal{R}}^{\gamma,q}(f)\|_{L^\infty(\mathbb{B}_n)} \approx \|A_{\tilde{\nabla}}^{\gamma,q}(f)\|_{L^\infty(\mathbb{B}_n)} \approx \|A_{\tilde{\nabla}}^{\gamma,q}(f)\|_{L^\infty(\mathbb{B}_n)},$$

where “ $\approx$ ” depends only on  $q, \gamma$ , and  $n$ .

### 3. ATOMIC DECOMPOSITION FOR BERGMAN SPACES

We let

$$d(z, w) = |1 - \langle z, w \rangle|^{\frac{1}{2}}, \quad z, w \in \overline{\mathbb{B}_n}.$$

It is known that  $d$  satisfies the triangle inequality and the restriction of  $d$  to  $\mathbb{S}_n$  is a metric. As usual,  $d$  is called the nonisotropic metric.

For any  $\zeta \in \mathbb{S}_n$  and  $r > 0$ , the set

$$Q_r(\zeta) = \{z \in \mathbb{B}_n : d(z, \zeta) < r\}$$

is called a Carleson tube with respect to the nonisotropic metric  $d$ . We usually write  $Q = Q_r(\zeta)$  in short.

As usual, we define the atoms with respect to the Carleson tube as follows: for  $1 < q < \infty$ ,  $a \in L^q(\mathbb{B}_n, dv_\alpha)$  is said to be a  $(1, q)_\alpha$ -atom if there is a Carleson tube  $Q$  such that

- (1)  $a$  is supported in  $Q$ ;

- (2)  $\|a\|_{L^q(\mathbb{B}_n, dv_\alpha)} \leq v_\alpha(Q)^{\frac{1}{q}-1};$   
 (3)  $\int_{\mathbb{B}_n} a(z) dv_\alpha(z) = 0.$

The constant function 1 is also considered to be a  $(1, q)_\alpha$ -atom.

Recall that  $P_\alpha$  is the orthogonal projection from  $L^2(\mathbb{B}_n, dv_\alpha)$  onto  $\mathcal{A}_\alpha^2$ , which can be expressed as

$$P_\alpha f(z) = \int_{\mathbb{B}_n} K^\alpha(z, w) f(w) dv_\alpha(w), \quad \forall f \in L^1(\mathbb{B}_n, dv_\alpha), \alpha > -1,$$

where

$$K^\alpha(z, w) = \frac{1}{(1 - \langle z, w \rangle)^{n+1+\alpha}}, \quad z, w \in \mathbb{B}_n.$$

$P_\alpha$  extends to a bounded projection from  $L^p(\mathbb{B}_n, dv_\alpha)$  onto  $\mathcal{A}_\alpha^p$  ( $1 < p < \infty$ ).

We have the following useful estimates.

**Lemma 3.1.** *For  $\alpha > -1$  and  $1 < q < \infty$  there exists a constant  $C_{q,\alpha} > 0$  such that*

$$\|P_\alpha(a)\|_{1,\alpha} \leq C_{q,\alpha}$$

for any  $(1, q)_\alpha$ -atom  $a$ .

To prove Lemma 3.1, we need first to show an inequality for reproducing kernel  $K^\alpha$  associated with  $d$ , which is essentially borrowed from Proposition 2.13 in [12].

**Lemma 3.2.** *For  $\alpha > -1$  there exists a constant  $\delta > 0$  such that for all  $z, w \in \mathbb{B}_n, \zeta \in \mathbb{S}_n$  satisfying  $d(z, \zeta) > \delta d(w, \zeta)$ , we have*

$$|K^\alpha(z, w) - K^\alpha(z, \zeta)| \leq C_{\alpha,n} \frac{d(w, \zeta)}{d(z, \zeta)^{2(n+1+\alpha)+1}}.$$

*Proof.* Note that

$$K^\alpha(z, w) - K^\alpha(z, \zeta) = \int_0^1 \frac{d}{dt} \left( \frac{1}{(1 - \langle z, \zeta \rangle - t \langle z, w - \zeta \rangle)^{n+1+\alpha}} \right) dt.$$

We have

$$|K^\alpha(z, w) - K^\alpha(z, \zeta)| \leq \int_0^1 \frac{(n+1+\alpha) |\langle z, w - \zeta \rangle|}{|1 - \langle z, \zeta \rangle - t \langle z, w - \zeta \rangle|^{n+2+\alpha}} dt.$$

Write  $z = z_1 + z_2$  and  $w = w_1 + w_2$ , where  $z_1$  and  $w_1$  are parallel to  $\zeta$ , while  $z_2$  and  $w_2$  are perpendicular to  $\zeta$ . Then

$$\langle z, w \rangle - \langle z, \zeta \rangle = \langle z_2, w_2 \rangle - \langle z_1, w_1 - \zeta \rangle$$

and so

$$|\langle z, w \rangle - \langle z, \zeta \rangle| \leq |z_2| |w_2| + |w_1 - \zeta|.$$

Since  $|w_1 - \zeta| = |1 - \langle w, \zeta \rangle|$ ,

$$\begin{aligned} |z_2|^2 &= |z|^2 - |z_1|^2 < 1 - |z_1|^2 < (1 + |z_1|)(1 + |z_1|) \\ &\leq |1 - \langle z_1, \zeta \rangle| = 2|1 - \langle z, \zeta \rangle|, \end{aligned}$$

and similarly

$$|w_2|^2 \leq 2|1 - \langle w, \zeta \rangle|,$$

we have

$$\begin{aligned} |\langle z, w \rangle - \langle z, \zeta \rangle| &\leq 2|1 - \langle z, \zeta \rangle|^{1/2}|1 - \langle w, \zeta \rangle|^{1/2} + |1 - \langle w, \zeta \rangle| \\ &= 2d(w, \zeta)[d(z, \zeta) + d(w, \zeta)] \\ &\leq 2\left(1 + \frac{1}{\delta}\right)\frac{1}{\delta}d^2(z, \zeta). \end{aligned}$$

This concludes that there is  $\delta > 1$  such that

$$|\langle z, w - \zeta \rangle| < \frac{1}{2}|1 - \langle z, \zeta \rangle|, \quad \forall z, w \in \mathbb{B}_n, \quad \zeta \in \mathbb{S}_n,$$

whenever  $d(z, \zeta) > \delta d(w, \zeta)$ . Then, we have

$$|1 - \langle z, \zeta \rangle - t\langle z, w - \zeta \rangle| > |1 - \langle z, \zeta \rangle| - t|\langle z, \zeta - w \rangle| > \frac{1}{2}|1 - \langle z, \zeta \rangle|.$$

Therefore,

$$\begin{aligned} |K^\alpha(z, w) - K^\alpha(z, \zeta)| &\leq \frac{2^{n+3+\alpha}(n+1+\alpha)(1+1/\delta)d(w, \zeta)d(z, \zeta)}{|1 - \langle z, \zeta \rangle|^{n+2+\alpha}} \\ &\leq C_{\alpha, n} \frac{d(w, \zeta)}{d(z, \zeta)^{2(n+1+\alpha)+1}} \end{aligned}$$

and the lemma is proved.  $\square$

*Proof of Lemma 3.1.* When  $a$  is the constant function 1, the result is clear. Thus we may suppose  $a$  is a  $(1, q)_\alpha$ -atom. Let  $a$  be supported in a Carleson tuber  $Q_r(\zeta)$  and  $\delta r \leq \sqrt{2}$ , where  $\delta$  is the constant in Lemma 3.2. Since  $P_\alpha$  is a bounded operator on  $L^q(\mathbb{B}_n, dv_\alpha)$ , we have

$$\begin{aligned} \int_{Q_{\delta r}} |P_\alpha(a)| dv_\alpha(z) &\leq v_\alpha(Q_{\delta r})^{1-\frac{1}{q}} \|P_\alpha(a)\|_{q, \alpha} \\ &\leq \|P_\alpha\|_{L^q(\mathbb{B}_n, dv_\alpha)} v_\alpha(Q_{\delta r})^{1-\frac{1}{q}} \|a\|_{q, \alpha} \\ &\leq \|P_\alpha\|_{L^q(\mathbb{B}_n, dv_\alpha)}. \end{aligned}$$

Next, if  $d(z, \zeta) > \delta r$  then

$$\begin{aligned}
& \left| \int_{\mathbb{B}_n} \frac{a(w)}{(1 - \langle z, w \rangle)^{n+1+\alpha}} dv_\alpha(w) \right| \\
&= \left| \int_{Q_r(\zeta)} a(w) \left[ \frac{1}{(1 - \langle z, w \rangle)^{n+1+\alpha}} - \frac{1}{(1 - \langle z, \zeta \rangle)^{n+1+\alpha}} \right] dv_\alpha(w) \right| \\
&\leq C \int_{Q_r(\zeta)} |a(w)| \frac{d(w, \zeta)}{d(z, \zeta)^{2(n+1+\alpha)+1}} dv_\alpha(w) \\
&\leq Cr \int_{Q_r(\zeta)} |a(w)| dv_\alpha(w) \frac{1}{d(z, \zeta)^{2(n+1+\alpha)+1}} \\
&\leq \frac{Cr}{d(z, \zeta)^{2(n+1+\alpha)+1}}.
\end{aligned}$$

Then

$$\begin{aligned}
& \int_{d(z, \zeta) > \delta r} |P_\alpha(a)| dv_\alpha(z) \\
&\leq Cr \int_{d(z, \zeta) > \delta r} \frac{1}{d(z, \zeta)^{2(n+1+\alpha)+1}} dv_\alpha(z) \\
&= Cr \sum_{k \geq 0} \int_{2^k \delta r < d(z, \zeta) \leq 2^{k+1} \delta r} \frac{1}{d(z, \zeta)^{2(n+1+\alpha)+1}} dv_\alpha(z) \\
&\leq Cr \sum_{k \geq 0} \frac{v_\alpha(Q_{2^{k+1} \delta r})}{(2^k \delta r)^{2(n+1+\alpha)+1}} \\
&\leq Cr \sum_{k=0}^{\infty} \frac{(2^{k+1} \delta r)^{2(n+1+\alpha)}}{(2^k \delta r)^{2(n+1+\alpha)+1}} \leq C,
\end{aligned}$$

where we have used the fact that  $v_\alpha(Q_r) \approx r^{2(n+1+\alpha)}$  in the third inequality (e.g., Corollary 5.24 in [15]). Thus, we get

$$\int_{\mathbb{B}_n} |P_\alpha(a)| dv_\alpha(z) = \int_{Q_{\delta r}} |P_\alpha(a)| dv_\alpha(z) + \int_{d(z, \zeta) > \delta r} |P_\alpha(a)| dv_\alpha(z) \leq C,$$

where  $C$  depends only on  $n$  and  $\alpha$ .  $\square$

Now we turn to the real-variable atomic decomposition of  $\mathcal{A}_\alpha^1$  ( $\alpha > -1$ ) with respect to the Carleson tubes. Note that for any  $(1, q)_\alpha$ -atom  $a$ ,

$$\|a\|_{1, \alpha} = \int_Q |a| dv_\alpha \leq v_\alpha(Q)^{1-1/q} \|a\|_{q, \alpha} \leq 1.$$

Then, we define  $\mathcal{A}_\alpha^{1, q}$  as the space of all  $f \in \mathcal{A}_\alpha^1$  which admits a decomposition

$$f = \sum_i \lambda_i P_\alpha a_i \quad \text{and} \quad \sum_i |\lambda_i| \leq C_q \|f\|_{1, \alpha},$$

where for each  $i$ ,  $a_i$  is an  $(1, q)_\alpha$ -atom and  $\lambda_i \in \mathbb{C}$  so that  $\sum_i |\lambda_i| < \infty$ . We equip this space with the norm

$$\|f\|_{\mathcal{A}_\alpha^{1,q}} = \inf \left\{ \sum_i |\lambda_i| : f = \sum_i \lambda_i P_\alpha a_i \right\}$$

where the infimum is taken over all decompositions of  $f$  described above.

It is easy to see that  $\mathcal{A}_\alpha^{1,q}$  is a Banach space. By Lemma 3.1 we have the contractive inclusion  $\mathcal{A}_\alpha^{1,q} \subset \mathcal{A}_\alpha^1$ . We will prove in what follows that these two spaces coincide. That establishes the “real-variable” atomic decomposition of the Bergman space  $\mathcal{A}_\alpha^1$ . In fact, we will show the remaining inclusion  $\mathcal{A}_\alpha^1 \subset \mathcal{A}_\alpha^{1,q}$  by duality.

**Theorem 3.1.** *Let  $1 < q < \infty$  and  $\alpha > -1$ . For every  $f \in \mathcal{A}_\alpha^1$  there exist a sequence  $\{a_i\}$  of  $(1, q)_\alpha$ -atoms and a sequence  $\{\lambda_i\}$  of complex numbers such that*

$$(3.1) \quad f = \sum_i \lambda_i P_\alpha a_i \quad \text{and} \quad \sum_i |\lambda_i| \leq C_q \|f\|_{1,\alpha}.$$

Moreover,

$$\|f\|_{1,\alpha} \approx \inf \sum_i |\lambda_i|$$

where the infimum is taken over all decompositions of  $f$  described above and “ $\approx$ ” depends only on  $\alpha$  and  $q$ .

Recall that the dual space of  $\mathcal{A}_\alpha^1$  is the Bloch space  $\mathcal{B}$  (we refer to [15] for details). The Banach dual of  $\mathcal{A}_\alpha^1$  can be identified with  $\mathcal{B}$  (with equivalent norms) under the integral pairing

$$\langle f, g \rangle_\alpha = \lim_{r \rightarrow 1^-} \int_{\mathbb{B}_n} f(rz) \overline{g(z)} dv_\alpha(z), \quad f \in \mathcal{A}_\alpha^1, g \in \mathcal{B}.$$

(e.g., see Theorem 3.17 in [15].)

In order to prove Theorem 3.1, we need the following result, which can be found in [6] (see also Theorem 5.25 in [15]).

**Lemma 3.3.** *Suppose  $\alpha > -1$  and  $1 \leq p < \infty$ . Then, for any  $f \in \mathcal{H}(\mathbb{B}_n)$ ,  $f$  is in  $\mathcal{B}$  if and only if there exists a constant  $C > 0$  depending only on  $\alpha$  and  $p$  such that*

$$\frac{1}{v_\alpha(Q_r(\zeta))} \int_{Q_r(\zeta)} |f - f_{\alpha, Q_r(\zeta)}|^p dv_\alpha \leq C$$

for all  $r > 0$  and all  $\zeta \in \mathbb{S}_n$ , where

$$f_{\alpha, Q_r(\zeta)} = \frac{1}{Q_r(\zeta)} \int_{Q_r(\zeta)} f(z) dv_\alpha(z).$$

Moreover,

$$\|f\|_{\mathcal{B}} \approx \sup_{r>0, \zeta \in \mathbb{S}} \left( \frac{1}{v_\alpha(Q_r(\zeta))} \int_{Q_r(\zeta)} |f - f_{\alpha, Q_r(\zeta)}|^p dv_\alpha \right)^{\frac{1}{p}},$$

where “ $\approx$ ” depends only on  $\alpha, p$ , and  $n$ .

As noted above, we will prove Theorem 3.1 via duality. To this end, we first prove the following duality theorem.

**Proposition 3.1.** *For any  $1 < q < \infty$  and  $\alpha > -1$ , we have  $(\mathcal{A}_\alpha^{1,q})^* = \mathcal{B}$  isometrically. More precisely,*

(i) *Every  $g \in \mathcal{B}$  defines a continuous linear functional  $\varphi_g$  on  $\mathcal{A}_\alpha^{1,q}$  by*

$$(3.2) \quad \varphi_g(f) = \lim_{r \rightarrow 1^-} \int_{\mathbb{B}_n} f(rz) \overline{g(z)} dv_\alpha(z), \quad \forall f \in \mathcal{A}_\alpha^{1,q}.$$

(ii) *Conversely, each  $\varphi \in (\mathcal{A}_\alpha^{1,q})^*$  is given as (3.2) by some  $g \in \mathcal{B}$ .*

Moreover, we have

$$(3.3) \quad \|\varphi_g\| \approx |g(0)| + \|g\|_{\mathcal{B}}, \quad \forall g \in \mathcal{B}.$$

*Proof.* Let  $p$  be the conjugate index of  $q$ , i.e.,  $1/p + 1/q = 1$ . We first show  $\mathcal{B} \subset (\mathcal{A}_\alpha^{1,q})^*$ . Let  $g \in \mathcal{B}$ . For any  $(1, q)_\alpha$ -atom  $a$ , by Lemma 3.3 we have

$$\begin{aligned} \left| \int_{\mathbb{B}_n} P_\alpha a(z) \overline{g(z)} dv_\alpha(z) \right| &= |\langle P_\alpha(a_j), g \rangle_\alpha| \\ &= \left| \int_{\mathbb{B}_n} a \bar{g} dv_\alpha \right| \\ &= \left| \int_{\mathbb{B}_n} a \overline{(g - g_Q)} dv_\alpha \right| \\ &\leq \left( \int_Q |a|^q dv_\alpha \right)^{1/q} \left( \int_Q |g - g_Q|^p dv_\alpha \right)^{1/p} \\ &\leq \left( \frac{1}{v_\alpha(Q)} \int_Q |g - g_Q|^p dv_\alpha \right)^{1/p} \\ &\leq C \|g\|_{\mathcal{B}}. \end{aligned}$$

On the other hand, for the constant function 1 we have  $P_\alpha 1 = 1$  and so

$$\left| \int_{\mathbb{B}_n} P_\alpha 1(z) \overline{g(z)} dv_\alpha(z) \right| = \left| \int_{\mathbb{B}_n} g(z) dv_\alpha(z) \right| = |g(0)|.$$

Thus, we deduce that

$$\left| \int_{\mathbb{B}_n} f \bar{g} dv_\alpha \right| \leq C \|f\|_{\mathcal{A}_\alpha^{1,q}} (|g(0)| + \|g\|_{\mathcal{B}})$$

for any finite linear combination  $f$  of  $(1, q)_\alpha$ -atoms. Hence,  $g$  defines a continuous linear functional  $\varphi_g$  on a dense subspace of  $\mathcal{A}_\alpha^{1,q}$  and  $\varphi_g$  extends to a continuous linear functional on  $\mathcal{A}_\alpha^{1,q}$  such that

$$|\varphi_g(f)| \leq C (|g(0)| + \|g\|_{\mathcal{B}}) \|f\|_{\mathcal{A}_\alpha^{1,q}}$$

for all  $f \in \mathcal{A}_\alpha^{1,q}$ .

Next let  $\varphi$  be a bounded linear functional on  $\mathcal{A}_\alpha^{1,q}$ . Note that

$$\mathcal{H}^q(\mathbb{B}_n, dv_\alpha) = \mathcal{H}(\mathbb{B}_n) \cap L^q(\mathbb{B}_n, dv_\alpha) \subset \mathcal{A}_\alpha^{1,q}.$$

Then,  $\varphi$  is a bounded linear functional on  $\mathcal{H}^q(\mathbb{B}_n, dv_\alpha)$ . By duality there exists  $g \in \mathcal{H}^p(\mathbb{B}_n, dv_\alpha)$  such that

$$\varphi(f) = \int_{\mathbb{B}_n} f \bar{g} dv_\alpha, \quad \forall f \in \mathcal{H}^q(\mathbb{B}_n, dv_\alpha).$$

Let  $Q = Q_r(\zeta)$  be a Carleson tube. For any  $f \in L^q(\mathbb{B}_n, dv_\alpha)$  supported in  $Q$ , it is easy to check that

$$a_f = (f - f_Q) \chi_Q / [\|f\|_{L^q v_\alpha(Q)}^{1/p}]$$

is a  $(1, q)$ -atom. Then,  $|\varphi(P_\alpha a_f)| \leq \|\varphi\|$  and so

$$|\varphi(P_\alpha[(f - f_Q) \chi_Q])| \leq \|\varphi\| \|f\|_{L^q v_\alpha(Q)}^{1/p}.$$

Hence, for any  $f \in L^q(\mathbb{B}_n, dv_\alpha)$  we have

$$\begin{aligned} \left| \int_Q \overline{f - f_Q} dv_\alpha \right| &= \left| \int_Q (f - f_Q) \bar{g} dv_\alpha \right| \\ &= \left| \int_{\mathbb{B}_n} (f - f_Q) \chi_Q \bar{g} dv_\alpha \right| \\ &= \left| \int_{\mathbb{B}_n} P_\alpha[(f - f_Q) \chi_Q] \bar{g} dv_\alpha \right| \\ &= |\varphi(P_\alpha[(f - f_Q) \chi_Q])| \\ &\leq \|\varphi\| \|(f - f_Q) \chi_Q\|_{L^q(\mathbb{B}_n, dv_\alpha)} v_\alpha(Q)^{1/p} \\ &\leq 2\|\varphi\| \|f\|_{L^q(Q, dv_\alpha)} v_\alpha(Q)^{1/p}. \end{aligned}$$

This concludes that

$$\left( \frac{1}{v_\alpha(Q)} \int_Q |g - g_Q|^p dv_\alpha \right)^{1/p} \leq 2\|\varphi\|.$$

By Lemma 3.3 we have that  $g \in \mathcal{B}$  and  $\|g\|_{\mathcal{B}} \leq C\|\varphi\|$ . Therefore,  $\varphi$  is given as (3.2) by  $g$  with  $|g(0)| + \|g\|_{\mathcal{B}} \leq C\|\varphi\|$ .  $\square$

Now we are ready to prove Theorem 3.1.

*Proof of Theorem 3.1.* By Lemma 3.1 we know that  $\mathcal{A}_\alpha^{1,q} \subset \mathcal{A}_\alpha^1$ . On the other hand, by Proposition 3.1 we have  $(\mathcal{A}_\alpha^1)^* = (\mathcal{A}_\alpha^{1,q})^*$ . Hence, by duality we have  $\|f\|_{1,q} \approx \|f\|_{\mathcal{A}_\alpha^{1,q}}$ .  $\square$

**Remark 3.1.** (1) One would like to expect that when  $0 < p < 1$ ,  $\mathcal{A}_\alpha^p$  also admits an atomic decomposition in terms of atoms with respect to Carleson tubes. However, the proof of Theorem 3.1 via duality cannot be extended to the case  $0 < p < 1$ . At the time of this writing, this problem is entirely open.



- (2) The real-variable atomic decomposition of Bergman spaces should be known to specialists, at least in the case  $p = 1$ . Indeed, Coifman and Weiss [9] claimed that the Bergman space  $\mathcal{A}^1$  admits an atomic decomposition, based on their theory of harmonic analysis on homogeneous spaces. However, the case  $0 < p < 1$  seems open as well.

#### 4. AREA INTEGRAL INEQUALITIES: REAL-VARIABLE METHODS

In this section, we will prove the area integral inequality for the Bergman space  $\mathcal{A}_\alpha^1$  via atomic decomposition established in Section 3.

**Theorem 4.1.** *Suppose  $1 < q < \infty$ ,  $\gamma > 0$ , and  $\alpha > -1$ . Then,*

$$(4.1) \quad \|A_{\tilde{\nabla}}^{\gamma,q}(f)\|_{1,\alpha} \lesssim \|f\|_{1,\alpha}, \quad \forall f \in \mathcal{H}(\mathbb{B}_n).$$

This is the assertion that (a)  $\Rightarrow$  (d) in Theorem 1.2 for the case  $p = 1$ . The novelty is that the proof we present here utilizes a real-variable method.

The following lemma is elementary.

**Lemma 4.1.** *Suppose  $1 < q < \infty$ ,  $\gamma > 0$  and  $\alpha > -1$ . If  $f \in \mathcal{A}_\alpha^q$ , then*

$$\int_{\mathbb{B}_n} |A_{\tilde{\nabla}}^{\gamma,q}(f)(z)|^q dv_\alpha \approx \int_{\mathbb{B}_n} |f(z) - f(0)|^q dv_\alpha,$$

where “ $\approx$ ” depends only on  $q, \gamma, \alpha$ , and  $n$ .

*Proof.* Note that  $v_\alpha(D(z, \gamma)) \approx (1 - |z|^2)^{n+1+\alpha}$ . Then

$$\begin{aligned} \int_{\mathbb{B}_n} |A_{\tilde{\nabla}}^{\gamma,q}(f)(z)|^q dv_\alpha &= \int_{\mathbb{B}_n} \int_{D(z, \gamma)} (1 - |w|^2)^{-1-n} |\tilde{\nabla} f(w)|^q dv(w) dv_\alpha(z) \\ &= \int_{\mathbb{B}_n} v_\alpha(D(w, \gamma)) (1 - |w|^2)^{-1-n} |\tilde{\nabla} f(w)|^q dv(w) \\ &\approx \int_{\mathbb{B}_n} |\tilde{\nabla} f(w)|^q dv_\alpha(w) \\ &\approx \int_{\mathbb{B}_n} |f(w) - f(0)|^q dv_\alpha(w). \end{aligned}$$

In the last step we have used Theorem 2.16 (b) in [15].  $\square$

*Proof of (4.1).* By Theorem 3.1, it suffices to show that for  $1 < q < \infty$ ,  $\gamma > 0$ , and  $\alpha > -1$  there exists  $C > 0$  such that

$$\|A_{\tilde{\nabla}}^{\gamma,q}(P_\alpha a)\|_{1,\alpha} \leq C$$

for all  $(1, q)_\alpha$ -atoms  $a$ . Given an  $(1, q)_\alpha$ -atom  $a$  supported in  $Q = Q_r(\zeta)$ . By Lemma 4.1 we have

$$\begin{aligned} \int_{2Q} A_{\tilde{\nabla}}^{\gamma, q}(P_\alpha a) dv_\alpha &\leq v_\alpha(2Q)^{1-\frac{1}{q}} \left( \int_{2Q} [A_{\tilde{\nabla}}^{\gamma, q}(P_\alpha a)]^q dv_\alpha \right)^{\frac{1}{q}} \\ &\leq C v_\alpha(Q)^{1-\frac{1}{q}} \left( \int_{\mathbb{B}_n} |P_\alpha a(z) - P_\alpha a(0)|^q dv_\alpha \right)^{\frac{1}{q}} \\ &\leq C v_\alpha(Q)^{1-\frac{1}{q}} \|a\|_{q, \alpha} \leq C, \end{aligned}$$

where  $2Q = Q_{2r}(\zeta)$ . On the other hand,

$$\begin{aligned} &\int_{(2Q)^c} A_{\tilde{\nabla}}^{\gamma, q}(P_\alpha a) dv_\alpha \\ &= \int_{(2Q)^c} \left( \int_{D(z, \gamma)} |\tilde{\nabla} P_\alpha a(w)|^q d\tau(w) \right)^{\frac{1}{q}} dv_\alpha(z) \\ &= \int_{(2Q)^c} \left( \int_{D(z, \gamma)} \left| \int_Q \tilde{\nabla}_w [K^\alpha(w, u) - K^\alpha(w, \zeta)] a(u) dv_\alpha(u) \right|^q d\tau(w) \right)^{\frac{1}{q}} dv_\alpha(z) \\ &\leq \|a\|_{q, \alpha} \int_{(2Q)^c} \left( \int_{D(z, \gamma)} \left( \int_Q |\tilde{\nabla}_w [K^\alpha(w, u) - K^\alpha(w, \zeta)]|^{\frac{q}{q-1}} dv_\alpha(u) \right)^{q-1} d\tau(w) \right)^{\frac{1}{q}} dv_\alpha(z) \\ &\leq \int_{(2Q)^c} \left( \int_{D(z, \gamma)} \sup_{u \in Q} |\tilde{\nabla}_w [K^\alpha(w, u) - K^\alpha(w, \zeta)]|^q d\tau(w) \right)^{\frac{1}{q}} dv_\alpha(z), \end{aligned}$$

where  $(2Q)^c = \mathbb{B}_n \setminus 2Q$ .

An immediate computation yields that

$$\begin{aligned} &\nabla_w [K^\alpha(w, u) - K^\alpha(w, \zeta)] \\ &= (n+1+\alpha) \left[ \frac{\bar{u}}{(1-\langle w, u \rangle)^{n+2+\alpha}} - \frac{\bar{\zeta}}{(1-\langle w, \zeta \rangle)^{n+2+\alpha}} \right] \\ &= (n+1+\alpha) \frac{\bar{u}(1-\langle w, \zeta \rangle)^{n+2+\alpha} - \bar{\zeta}(1-\langle w, u \rangle)^{n+2+\alpha}}{(1-\langle w, u \rangle)^{n+2+\alpha}(1-\langle w, \zeta \rangle)^{n+2+\alpha}} \end{aligned}$$

and

$$\begin{aligned} &\mathcal{R}_w [K^\alpha(w, u) - K^\alpha(w, \zeta)] \\ &= (n+1+\alpha) \left[ \frac{\langle w, u \rangle}{(1-\langle w, u \rangle)^{n+2+\alpha}} - \frac{\langle w, \zeta \rangle}{(1-\langle w, \zeta \rangle)^{n+2+\alpha}} \right] \\ &= (n+1+\alpha) \frac{\langle w, u \rangle(1-\langle w, \zeta \rangle)^{n+2+\alpha} - \langle w, \zeta \rangle(1-\langle w, u \rangle)^{n+2+\alpha}}{(1-\langle w, u \rangle)^{n+2+\alpha}(1-\langle w, \zeta \rangle)^{n+2+\alpha}}. \end{aligned}$$

Moreover,

$$\begin{aligned}
& \left| \nabla_w [K^\alpha(w, u) - K^\alpha(w, \zeta)] \right|^2 \\
&= (n+1+\alpha)^2 \left\{ \frac{|u|^2 |1 - \langle w, \zeta \rangle|^{2(n+2+\alpha)} + |1 - \langle w, u \rangle|^{2(n+2+\alpha)}}{|1 - \langle w, u \rangle|^{2(n+2+\alpha)} |1 - \langle w, \zeta \rangle|^{2(n+2+\alpha)}} \right. \\
&\quad - \frac{(1 - \langle w, \zeta \rangle)^{n+2+\alpha} (1 - \langle u, w \rangle)^{n+2+\alpha} \langle \zeta, u \rangle}{|1 - \langle w, u \rangle|^{2(n+2+\alpha)} |1 - \langle w, \zeta \rangle|^{2(n+2+\alpha)}} \\
&\quad \left. - \frac{(1 - \langle w, u \rangle)^{n+2+\alpha} (1 - \langle \zeta, w \rangle)^{n+2+\alpha} \langle u, \zeta \rangle}{|1 - \langle w, u \rangle|^{2(n+2+\alpha)} |1 - \langle w, \zeta \rangle|^{2(n+2+\alpha)}} \right\},
\end{aligned}$$

and

$$\begin{aligned}
& \left| \mathcal{R}_w [K^\alpha(w, u) - K^\alpha(w, \zeta)] \right|^2 \\
&= (n+1+\alpha)^2 \left\{ \frac{|\langle w, u \rangle|^2 |1 - \langle w, \zeta \rangle|^{2(n+2+\alpha)} + |\langle w, \zeta \rangle|^2 |1 - \langle w, u \rangle|^{2(n+2+\alpha)}}{|1 - \langle w, u \rangle|^{2(n+2+\alpha)} |1 - \langle w, \zeta \rangle|^{2(n+2+\alpha)}} \right. \\
&\quad - \frac{\langle w, u \rangle \langle \zeta, w \rangle (1 - \langle w, \zeta \rangle)^{n+2+\alpha} (1 - \langle u, w \rangle)^{n+2+\alpha}}{|1 - \langle w, u \rangle|^{2(n+2+\alpha)} |1 - \langle w, \zeta \rangle|^{2(n+2+\alpha)}} \\
&\quad \left. - \frac{\langle w, \zeta \rangle \langle u, w \rangle (1 - \langle w, u \rangle)^{n+2+\alpha} (1 - \langle \zeta, w \rangle)^{n+2+\alpha}}{|1 - \langle w, u \rangle|^{2(n+2+\alpha)} |1 - \langle w, \zeta \rangle|^{2(n+2+\alpha)}} \right\}.
\end{aligned}$$

Then, we have

$$\begin{aligned}
& \left| \nabla_w [K^\alpha(w, u) - K^\alpha(w, \zeta)] \right|^2 - \left| \mathcal{R}_w [K^\alpha(w, u) - K^\alpha(w, \zeta)] \right|^2 \\
&= \frac{(n+1+\alpha)^2}{|1 - \langle w, u \rangle|^{2(n+2+\alpha)} |1 - \langle w, \zeta \rangle|^{2(n+2+\alpha)}} \\
&\quad \times \left\{ (|u|^2 - |\langle w, u \rangle|^2) |1 - \langle w, \zeta \rangle|^{2(n+2+\alpha)} \right. \\
&\quad + (1 - |\langle w, \zeta \rangle|^2) |1 - \langle w, u \rangle|^{2(n+2+\alpha)} \\
&\quad + (\langle w, u \rangle \langle \zeta, w \rangle - \langle \zeta, u \rangle) (1 - \langle w, \zeta \rangle)^{n+2+\alpha} (1 - \langle u, w \rangle)^{n+2+\alpha} \\
&\quad \left. + (\langle w, \zeta \rangle \langle u, w \rangle - \langle u, \zeta \rangle) (1 - \langle w, u \rangle)^{n+2+\alpha} (1 - \langle \zeta, w \rangle)^{n+2+\alpha} \right\}.
\end{aligned}$$

Note that for any  $f \in \mathcal{H}(\mathbb{B}_n)$ ,

$$|\tilde{\nabla} f(z)|^2 = (1 - |z|^2)(|\nabla f(z)|^2 - |\mathcal{R}f(z)|^2), \quad z \in \mathbb{B}_n.$$

(e.g., Lemma 2.13 in [15].) It is concluded that

$$\begin{aligned}
& |\tilde{\nabla}_w[K^\alpha(w, u) - K^\alpha(w, \zeta)]| \\
& \leq \frac{(n+1+\alpha)(1-|w|^2)^{\frac{1}{2}}}{|1-\langle w, u \rangle|^{n+2+\alpha}|1-\langle w, \zeta \rangle|^{n+2+\alpha}} \\
& \quad \times \left\{ (1-|\langle w, u \rangle|^2)|1-\langle w, \zeta \rangle|^{2(n+2+\alpha)} \right. \\
& \quad + (1-|\langle w, \zeta \rangle|^2)|1-\langle w, u \rangle|^{2(n+2+\alpha)} \\
& \quad + [(\langle w, u - \zeta \rangle \langle \zeta, w \rangle + (|\langle w, \zeta \rangle|^2 - 1) + (1 - \langle \zeta, u \rangle))] \\
& \quad \times (1 - \langle w, \zeta \rangle)^{n+2+\alpha} (1 - \langle u, w \rangle)^{n+2+\alpha} \\
& \quad + [(\langle w, \zeta - u \rangle \langle u, w \rangle + (|\langle w, u \rangle|^2 - 1) + (1 - \langle u, \zeta \rangle))] \\
& \quad \times (1 - \langle w, u \rangle)^{n+2+\alpha} (1 - \langle \zeta, w \rangle)^{n+2+\alpha} \Big\}^{\frac{1}{2}} \\
& \leq \frac{(n+1+\alpha)(1-|w|^2)^{\frac{1}{2}}(M_1 + M_2 + M_3 + M_4)^{\frac{1}{2}}}{|1-\langle w, u \rangle|^{n+2+\alpha}|1-\langle w, \zeta \rangle|^{n+2+\alpha}},
\end{aligned}$$

where

$$\begin{aligned}
M_1 &= |1 - \langle w, \zeta \rangle|^{n+2+\alpha} |1 - \langle u, w \rangle|^{n+2+\alpha} \\
& \quad \times |\langle w, u - \zeta \rangle \langle \zeta, w \rangle + (1 - \langle \zeta, u \rangle)|, \\
M_2 &= |1 - \langle w, u \rangle|^{n+2+\alpha} |1 - \langle \zeta, w \rangle|^{n+2+\alpha} \\
& \quad \times |\langle w, \zeta - u \rangle \langle u, w \rangle + (1 - \langle u, \zeta \rangle)|, \\
M_3 &= (1 - |\langle w, u \rangle|^2) |1 - \langle \zeta, w \rangle|^{n+2+\alpha} \\
& \quad \times |(1 - \langle w, \zeta \rangle)^{n+2+\alpha} - (1 - \langle w, u \rangle)^{n+2+\alpha}|, \\
M_4 &= (1 - |\langle w, \zeta \rangle|^2) |1 - \langle u, w \rangle|^{n+2+\alpha} \\
& \quad \times |(1 - \langle w, u \rangle)^{n+2+\alpha} - (1 - \langle w, \zeta \rangle)^{n+2+\alpha}|,
\end{aligned}$$

for  $w \in D(z, \gamma)$ ,  $u \in Q_r(\zeta)$  and  $z \in \mathbb{B}_n, \zeta \in \mathbb{S}_n$ .

Hence,

$$\begin{aligned}
& \int_{(2Q)^c} A_{\tilde{\nabla}}^{\gamma, q}(P_\alpha a) dv_\alpha \\
& \leq \int_{(2Q)^c} \left( \int_{D(z, \gamma)} \sup_{u \in Q} |\tilde{\nabla}_w[K^\alpha(w, u) - K^\alpha(w, \zeta)]|^q d\tau(w) \right)^{\frac{1}{q}} dv_\alpha(z) \\
& \leq (n+1+\alpha) \int_{(2Q)^c} (I_1 + I_2 + I_3 + I_4) dv_\alpha(z),
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \left( \int_{D(z, \gamma)} \sup_{u \in Q} \frac{(1 - |w|^2)^{\frac{q}{2}} M_1^{\frac{q}{2}}}{|1 - \langle w, u \rangle|^{q(n+2+\alpha)} |1 - \langle w, \zeta \rangle|^{q(n+2+\alpha)}} d\tau(w) \right)^{\frac{1}{q}}, \\
I_2 &= \left( \int_{D(z, \gamma)} \sup_{u \in Q} \frac{(1 - |w|^2)^{\frac{q}{2}} M_2^{\frac{q}{2}}}{|1 - \langle w, u \rangle|^{q(n+2+\alpha)} |1 - \langle w, \zeta \rangle|^{q(n+2+\alpha)}} d\tau(w) \right)^{\frac{1}{q}}, \\
I_3 &= \left( \int_{D(z, \gamma)} \sup_{u \in Q} \frac{(1 - |w|^2)^{\frac{q}{2}} M_3^{\frac{q}{2}}}{|1 - \langle w, u \rangle|^{q(n+2+\alpha)} |1 - \langle w, \zeta \rangle|^{q(n+2+\alpha)}} d\tau(w) \right)^{\frac{1}{q}}, \\
I_4 &= \left( \int_{D(z, \gamma)} \sup_{u \in Q} \frac{(1 - |w|^2)^{\frac{q}{2}} M_4^{\frac{q}{2}}}{|1 - \langle w, u \rangle|^{q(n+2+\alpha)} |1 - \langle w, \zeta \rangle|^{q(n+2+\alpha)}} d\tau(w) \right)^{\frac{1}{q}}.
\end{aligned}$$

We first estimate  $I_1$ . Note that

$$\begin{aligned}
M_1 &\leq (|\langle w, u - \zeta \rangle| + |1 - \langle \zeta, u \rangle|) |1 - \langle w, \zeta \rangle|^{n+2+\alpha} |1 - \langle w, u \rangle|^{n+2+\alpha} \\
&\leq \left( 2|1 - \langle u, \zeta \rangle|^{\frac{1}{2}} (|1 - \langle w, \zeta \rangle|^{\frac{1}{2}} + |1 - \langle u, \zeta \rangle|^{\frac{1}{2}}) + |1 - \langle \zeta, u \rangle| \right) \\
&\quad \times |1 - \langle w, \zeta \rangle|^{n+2+\alpha} |1 - \langle w, u \rangle|^{n+2+\alpha} \\
&\leq \left( 2|1 - \langle u, \zeta \rangle|^{\frac{1}{2}} (C_\gamma |1 - \langle z, \zeta \rangle|^{\frac{1}{2}} + \frac{1}{2} |1 - \langle z, \zeta \rangle|^{\frac{1}{2}}) + |1 - \langle \zeta, u \rangle| \right) \\
&\quad \times |1 - \langle w, \zeta \rangle|^{n+2+\alpha} |1 - \langle w, u \rangle|^{n+2+\alpha} \\
&\leq \left( C_\gamma r |1 - \langle z, \zeta \rangle|^{\frac{1}{2}} + r^2 \right) |1 - \langle w, \zeta \rangle|^{n+2+\alpha} |1 - \langle w, u \rangle|^{n+2+\alpha},
\end{aligned}$$

where the second inequality is the consequence of the following fact which has appeared in the proof of Lemma 3.2

$$|\langle w, u - \zeta \rangle| \leq 2|1 - \langle u, \zeta \rangle|^{\frac{1}{2}} (|1 - \langle w, \zeta \rangle|^{\frac{1}{2}} + |1 - \langle u, \zeta \rangle|^{\frac{1}{2}});$$

the third inequality is obtained by Lemma 2.3 and the fact

$$|1 - \langle u, \zeta \rangle|^{\frac{1}{2}} < r < \frac{1}{2} |1 - \langle z, \zeta \rangle|^{\frac{1}{2}}$$

for  $u \in Q$  and  $z \in (2Q)^c$ . Since

$$\begin{aligned}
|1 - \langle z, u \rangle|^{\frac{1}{2}} &\geq |1 - \langle z, \zeta \rangle|^{\frac{1}{2}} - |1 - \langle u, \zeta \rangle|^{\frac{1}{2}} \\
&\geq |1 - \langle z, \zeta \rangle|^{\frac{1}{2}} - \frac{1}{2} |1 - \langle z, \zeta \rangle|^{\frac{1}{2}} \\
&\geq \frac{1}{2} |1 - \langle z, \zeta \rangle|^{\frac{1}{2}},
\end{aligned}$$

by Lemmas 2.1 and 2.3 we have

$$\begin{aligned}
I_1 &\leq \left( \int_{D(z, \gamma)} \sup_{u \in Q} \frac{(1 - |w|^2)^{\frac{q}{2}} [Cr|1 - \langle z, \zeta \rangle|^{\frac{1}{2}} + r^2]^{\frac{q}{2}}}{|1 - \langle w, u \rangle|^{\frac{q}{2}(n+2+\alpha)} |1 - \langle w, \zeta \rangle|^{\frac{q}{2}(n+2+\alpha)}} d\tau(w) \right)^{\frac{1}{q}} \\
&\leq C_\gamma \left( \int_{D(z, \gamma)} \sup_{u \in Q} \frac{(1 - |z|^2)^{\frac{q}{2}} (Cr|1 - \langle z, \zeta \rangle|^{\frac{1}{2}} + r^2)^{\frac{q}{2}}}{|1 - \langle z, u \rangle|^{\frac{q}{2}(n+2+\alpha)} |1 - \langle z, \zeta \rangle|^{\frac{q}{2}(n+2+\alpha)}} d\tau(w) \right)^{\frac{1}{q}} \\
&\leq C_\gamma \left( \frac{(1 - |z|^2)^{\frac{q}{2}} (r|1 - \langle z, \zeta \rangle|^{\frac{1}{2}} + r^2)^{\frac{q}{2}}}{|1 - \langle z, \zeta \rangle|^{q(n+2+\alpha)}} \right)^{\frac{1}{q}} \\
&\leq C_\gamma \frac{(r|1 - \langle z, \zeta \rangle|^{\frac{1}{2}} + r^2)^{\frac{1}{2}}}{|1 - \langle z, \zeta \rangle|^{n+\frac{3}{2}+\alpha}} \\
&\leq C_\gamma \left( \frac{r^{\frac{1}{2}}}{d(z, \zeta)^{2(n+1+\alpha)+\frac{1}{2}}} + \frac{r}{d(z, \zeta)^{2(n+1+\alpha)+1}} \right).
\end{aligned}$$

Hence,

$$\begin{aligned}
\int_{(2Q)^c} I_1 dv_\alpha(z) &\lesssim \int_{(2Q)^c} \frac{r^{\frac{1}{2}}}{d(z, \zeta)^{2(n+1+\alpha)+\frac{1}{2}}} dv_\alpha(z) \\
&\quad + \int_{(2Q)^c} \frac{r}{d(z, \zeta)^{2(n+1+\alpha)+1}} dv_\alpha(z).
\end{aligned}$$

The second term on the right hand side has been estimated in the proof of Lemma 3.1. The first term can be estimated as follows:

$$\begin{aligned}
&\int_{d(z, \zeta) > 2r} \frac{r^{1/2}}{d(z, \zeta)^{2(n+1+\alpha)+1/2}} dv_\alpha(z) \\
&= r^{1/2} \sum_{k \geq 0} \int_{2^k r < d(z, \zeta) \leq 2^{k+1} r} \frac{1}{d(z, \zeta)^{2(n+1+\alpha)+1/2}} dv_\alpha(z) \\
&\leq r^{1/2} \sum_{k \geq 0} \frac{v_\alpha(Q_{2^{k+1}r})}{(2^k r)^{2(n+1+\alpha)+1/2}} \\
&\leq C r^{1/2} \sum_{k=0}^{\infty} \frac{(2^{k+1}r)^{2(n+1+\alpha)}}{(2^k r)^{2(n+1+\alpha)+1/2}} \leq C,
\end{aligned}$$

where we have used the fact that  $v_\alpha(Q_r) \approx r^{2(n+1+\alpha)}$  in the third inequality (e.g., Corollary 5.24 in [15]).

By the same argument we can estimate  $I_2$  and omit the details.

Next, we estimate  $I_3$ . Note that

$$\begin{aligned}
M_3 &\leq (1 - |\langle w, u \rangle|^2) |1 - \langle w, \zeta \rangle|^{n+2+\alpha} \\
&\quad \times |(1 - \langle w, \zeta \rangle)^{n+2+\alpha} - (1 - \langle w, u \rangle)^{n+2+\alpha}| \\
&\leq 2 |1 - \langle w, u \rangle| |1 - \langle w, \zeta \rangle|^{n+2+\alpha} \\
&\quad \times \left| \int_0^1 \frac{d}{dt} (1 - \langle w, t\zeta + (1-t)u \rangle)^{n+2+\alpha} dt \right| \\
&= 2(n+2+\alpha) |1 - \langle w, u \rangle| |1 - \langle w, \zeta \rangle|^{n+2+\alpha} \\
&\quad \times \left| \langle w, \zeta - u \rangle \int_0^1 (1 - \langle w, t\zeta + (1-t)u \rangle)^{n+1+\alpha} dt \right| \\
&\leq C_\gamma |1 - \langle w, u \rangle| |1 - \langle w, \zeta \rangle|^{n+2+\alpha} r |1 - \langle z, \zeta \rangle|^{n+3/2+\alpha},
\end{aligned}$$

where the last inequality is achieved by the following estimates

$$\begin{aligned}
|1 - \langle w, t\zeta + (1-t)u \rangle| &\leq C_\gamma |1 - \langle z, t\zeta + (1-t)u \rangle| \\
&\leq C_\gamma |1 - \langle z, u \rangle| + |\langle z, \zeta - u \rangle| \\
&\leq C_\gamma |1 - \langle z, \zeta \rangle|
\end{aligned}$$

and

$$|\langle w, \zeta - u \rangle| \leq C_\gamma r |1 - \langle z, \zeta \rangle|^{\frac{1}{2}},$$

for any  $w \in D(z, \gamma)$  and  $u \in Q_r(\zeta)$ . Thus, by Lemmas 2.1 and 2.3

$$\begin{aligned}
I_3 &\leq C_\gamma \left( \int_{D(z, \gamma)} \sup_{u \in Q} \frac{(1 - |w|^2)^{\frac{q}{2}} r^{\frac{q}{2}} |1 - \langle z, \zeta \rangle|^{\frac{q}{2}(n+\frac{3}{2}+\alpha)}}{|1 - \langle w, u \rangle|^{q(n+1+\alpha)+\frac{q}{2}} |1 - \langle w, \zeta \rangle|^{\frac{q}{2}(n+2+\alpha)}} d\tau(w) \right)^{\frac{1}{q}} \\
&\leq C_\gamma \left( \int_{D(z, \gamma)} \sup_{u \in Q} \frac{(1 - |z|^2)^{\frac{q}{2}} r^{\frac{q}{2}} |1 - \langle z, \zeta \rangle|^{\frac{q}{2}(n+\frac{3}{2}+\alpha)}}{|1 - \langle z, u \rangle|^{q(n+1+\alpha)+\frac{q}{2}} |1 - \langle z, \zeta \rangle|^{\frac{q}{2}(n+2+\alpha)}} d\tau(w) \right)^{\frac{1}{q}} \\
&\leq C_\gamma \left( \frac{(1 - |z|^2)^{\frac{q}{2}} r^{\frac{q}{2}}}{|1 - \langle z, \zeta \rangle|^{q(n+1+\alpha)+\frac{3}{4}q}} \right)^{\frac{1}{q}} \\
&\leq C_\gamma \frac{r^{\frac{1}{2}}}{d(z, \zeta)^{2(n+1+\alpha)+\frac{1}{2}}}.
\end{aligned}$$

Hence,

$$\int_{(2Q)^c} I_3 dv_\alpha(z) \leq C_\gamma \int_{(2Q)^c} \frac{r^{\frac{1}{2}}}{d(z, \zeta)^{2(n+1+\alpha)+\frac{1}{2}}} dv_\alpha(z) \leq C_\gamma,$$

as shown above.

Similarly, we can estimate  $I_4$  and omit the details. Therefore, combining above estimates we conclude that

$$\int_{(2Q)^c} A_\gamma(\tilde{\nabla} P_\alpha a) dv_\alpha \leq C,$$

where  $C$  depends only on  $q, \gamma, n$ , and  $\alpha$ . □

**Remark 4.1.** We remark that whenever  $\mathcal{A}_\alpha^p$  have an atomic decomposition in terms of atoms with respect to Carleson tubes for  $0 < p < 1$ , the argument of Theorem 1.2 works as well in this case. However, as noted in Remark 3.1, the problem of the atomic decomposition of  $\mathcal{A}_\alpha^p$  with respect to Carleson tubes for  $0 < p < 1$  is entirely open.

**Remark 4.2.** The area integral inequality in case  $1 < p < \infty$  can be also proved through using the method of vector-valued Calderón-Zygmund operators for Bergman spaces. This has been done in [7].

**Acknowledgement.** This research was supported in part by the NSFC under Grant No. 11171338.

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